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ON SOME CASES WHEN THE QUINTIC IS SOLVABLE BY ELEMENTARY METHODS.

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Abel has shown that the general equation of the fifth degree is not solvable by the ordinary method of extraction of roots. Abelian equations and cyclical equations in which the *roots* satisfy a certain condition can be so solved. It would be of value to have the necessary and sufficient condition which must be satisfied by the *coefficients*, instead of by the roots, in order that the given equation be an Abelian, or a cyclical, or, in fact, be solvable by root extraction, so that one could immediately tell, given an equation, whether it were solvable. I believe no one has as yet deduced such a condition for the general quintic.

It is the object of this paper to give, not the general condition indeed, but one which covers many special cases, in the hope that it may prove of interest or of value in further study of the subject.

Let the general quintic be

$$f(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0 \dots\dots\dots (I),$$

and let it be divisible without a remainder by

$$\varphi(x) = x^2 + px + q = 0.$$

Now divide $f(x)$ by $\varphi(x)$ and let the result be

$$\frac{f(x)}{\varphi(x)} = \psi(x) + \frac{F_1(p, q)x + F_2(p, q)}{\varphi(x)}$$

where

$$\psi(x) = x^3 + A_1 x^2 + A_2 x + A_3 = 0,$$

A_1, A_2 and A_3 being functions of p, q, a_1, a_2 , and a_3 , and where F_1 and F_2 arranged according to q are

$$F_1 = q^2(a_1 - 2p) + q(-a_3 + a_2 p - a_1 p^2 + p^3) + a_5 = 0 \dots\dots\dots (II)$$

and

$$F_2 = q^2 + q(-a_2 + 2a_1 p - 3p^2) + a_4 - a_3 p + a_2 p^2 - a_1 p^3 + p^4 = 0 \dots\dots (III).$$

Now if (II) and (III) were considered simultaneous and solved for p and q , they would give all the ten values each of p and q which would make $\varphi = 0$ divide $f(x) = 0$ without a remainder, but this would involve the solution of an equation of the tenth degree.

If, however, (II) and (III) be considered identical and the absolute terms and the coefficients of q in the two be equated after dividing (II) by $a_1 - 2p$, there results two equations in p , which must possess at least one common root.

These two equations in p are

$$5p^3 - 6a_1 p^2 + (2a_1^2 + a_2)p + a_3 - a_1 a_2 = 0 \dots\dots\dots (IV),$$

and

$$2p^5 - 3a_1 p^4 + (2a_2 + a_1^2)p^3 - (2a_3 + a_1 a_2)p^2 + (2a_4 + a_1 a_3)p + a_5 - a_1 a_4 = 0 \dots (V).$$

These equations (IV) and (V) will have a common root when their resultant, R , formed by Sylvester's method, is equal to zero, that is, when

$$\begin{aligned} R = & \begin{vmatrix} 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 & 0 & 0 & 0 \\ 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 & 0 & 0 \\ 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 & 0 \\ 0 & 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 \\ 0 & 0 & 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 \\ 2 & -3a_1 & a_1^2 + 2a_2 & -2a_3 - a_2 a_1 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 & 0 & 0 \\ 0 & 2 & -3a_1 & a_1^2 + 2a_2 & -2a_3 - a_1 a_2 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 & 0 \\ 0 & 0 & 2 & -3a_1 & a_1^2 + 2a_2 & -2a_3 - a_1 a_2 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 \end{vmatrix} \\ & = 0 \dots\dots\dots (VI). \end{aligned}$$

This is therefore the condition that the general quintic be solvable by considering (II) and (III) identical.

The value of the common root p is easily found by differentiating R , and this value of p substituted in (II) or (III) gives in general two values of q corresponding. The value of p common to (IV) and (V) with either of the corresponding values of q substituted in $\varphi = 0 = x^2 + px + q$ gives two roots of the proposed equation (I). Two more roots are given by taking the same value of p with the

other corresponding value of q .

We see that the condition $R=0$ among the coefficients requires that, for some one value of p , there shall be two values of q , *i. e.* that, for example, there exist among the roots such a relation as

$$x_1 + x_3 = x_2 + x_4$$

with

$$x_1 x_3 = x_2 x_4,$$

where x_0, x_1, x_2, x_3, x_4 represent the five roots. The other root, the fifth, is then given by $x_0 = 2p - a_1$.

In case of equal roots, so that for a known value of p the two values of q are equal, or in any case when it is desirable or easier, three of the roots can be found from $\psi(x)=0$ when any pair of corresponding values of p and q are known.

In this discussion no restriction is placed upon the character of the values of $a_1, a_2, \dots, a_5$ or p or q . They may be complex.

Whenever, therefore, the coefficients of a quintic satisfy the condition (VI), then p can have at most only nine values (instead of ten as in the general case) and the equation is solvable by this method. q may at the same time have ten values.

Example: Let the given equation be

$$x^5 - 10x^4 + 17x^3 + 76x^2 - 228x + 144 = 0.$$

In this case the condition (VI) is satisfied and equations (IV) and (V) have a single common root, $p = -3$. Then from (II) or (III), we have

$$q^2 + 16q - 36 = 0,$$

from which $q = 2$ or -18 . Therefore

$$x^2 - 3x + 2 = 0, \text{ from which } x = 1 \text{ or } 2.$$

$$\text{or } x^2 - 3x - 18 = 0, \text{ from which } x = -3 \text{ or } 6,$$

and finally $x = 2p - a_1 = -6 + 10 = 4$, and the roots are 1, 2, -3 , 4 and 6.

Urbana, Ill., December 27, 1897.